Dispersion and cluster waves in periodic media

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We study the propagation of time-harmonic waves governed by the Helmholtz equation in an infinite medium containing a periodic array of small inclusions of arbitrary shape with Dirichlet or transmission conditions at their interfaces. The inclusion size a is much smaller than the array period, and the wavelength is fixed. We show that the character of wave propagation depends on whether the Bloch vector $\boldsymbol{k}$ is simple or its multiplicity is greater than one. In both cases, the perturbation to the solution is proportional to the size of the inclusion. However, the character of the solution and its dispersion are completely different.

In the Dirichlet problem for a simple Bloch vector, the dispersion relation and the low frequency cutoff ω_c are given by

$$k^2 = |\boldsymbol{k}|^2 + \frac{4\pi a q}{|\Pi|} + \mathcal{O}(a^2), \quad \omega_c = 2c\sqrt{\frac{\pi a q}{|\Pi|}}\left(1 + \mathcal{O}(a)\right), \quad a \to 0, \tag{1}$$

where k is the wave number, c is the speed of wave propagation in the host medium, $|\Pi|$ is the volume of the cell of periodicity, and the coefficient q the capacity of the cavity ($q = 1$ and a is the radius for a sphere).

We derive the dispersion relation in the transmission problem. In particular, for the spherical inclusions it has the form

$$\omega = c_{\text{eff}}|\boldsymbol{k}| + \mathcal{O}(a^4), \quad c_{\text{eff}} = \sqrt{\frac{\langle\nu\rangle}{\overline{\gamma}}}, \quad \overline{\gamma} = \gamma_+(1-f) + \gamma_- f, \quad \langle\nu\rangle = \nu_+\left(1 + 3\,\frac{\nu_- - \nu_+}{\nu_- + 2\nu_+}\,f\right), \tag{2}$$

where f is the volume fraction of the inclusions $(-)$, γ is the average adiabatic bulk compressibility modulus of the medium $(+)$, and the average specific volume $\langle\nu\rangle$ is determined by Maxwell's formula. Formulas (2) explain why there is a huge reduction of the group velocity of acoustic waves propagating in bubbly water.

If the Bloch vector $\boldsymbol{k}$ has multiplicity $n \geqslant 2$ then solutions in the Dirichlet problem have the form of clusters u_s of perturbed plane waves propagating in directions $\boldsymbol{k} - \boldsymbol{m}_j,\ 1 \leqslant j \leqslant n$. One of the solutions and its dispersion relation has the form

$$u(\boldsymbol{x}) = \sum_{j=1}^{n} \mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{m}_j)\cdot\boldsymbol{x}} + \mathcal{O}(a), \quad k_1^2 = |\boldsymbol{k}|^2 + \frac{4\pi a q n}{|\Pi|} + O(a^2), \tag{3}$$

where $\boldsymbol{m}_j$ are the points of the reciprocal lattice such that $|\boldsymbol{k}| = |\boldsymbol{k} - \boldsymbol{m}_j|,\ 1 \leqslant j \leqslant n$.

In the transmission problem, Bloch vectors with multiplicity $n \geqslant 2$ give rise to the cluster waves containing n linearly independent waves. The solutions have the form of

$$u_s(\boldsymbol{x}) = \sum_{j=1}^{n} \mu_{j,s}\mathrm{e}^{-\mathrm{i}(\boldsymbol{k}-\boldsymbol{m}_j)\cdot\boldsymbol{x}} + \mathcal{O}(a), \tag{4}$$

where $\mu_{j,s}$ are components of the eigenvectors $\boldsymbol{\mu}_s = (\mu_{1,s}, \mu_{2,s}, \ldots, \mu_{n,s})$ of the matrix $\boldsymbol{M}$ which in the case of spherical inclusions has the form

$$M_{i,j} = \alpha + \beta\,\hat{\boldsymbol{k}}_i \cdot \hat{\boldsymbol{k}}_j, \quad 1 \leqslant i,j \leqslant n, \quad \alpha = 1 - \frac{\gamma_-}{\gamma_+}, \quad \beta = 3\,\frac{\varrho_+ - \varrho_-}{\varrho_+ + 2\varrho_-}, \tag{5}$$

where ϱ is the density and $\hat{\boldsymbol{k}}_i$ is the unit vector in the direction of $\boldsymbol{k}_i = \boldsymbol{k} - \boldsymbol{m}_i$.

References

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