Global and local gaps in periodic media

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We consider acoustic wave propagation through a periodic array of small inclusions of arbitrary shape. The inclusion size a is much smaller than the array period, while the wavelength is fixed. This problem is a singular perturbation of the problem without inclusions, and its solution does not converge to an unperturbed wave when $a \to 0$. However, the dispersion surface is close to the one for the inclusionless problem, and this implies the absence of global gaps in any fixed interval $(\varepsilon, \varepsilon^{-1})$ of the time frequency ω if a is small enough. The notion of local gaps, which depends on the choice of a wave vector $\boldsymbol{k}$, will be introduced and studied.

The amplitude u of the waves is governed by the equation

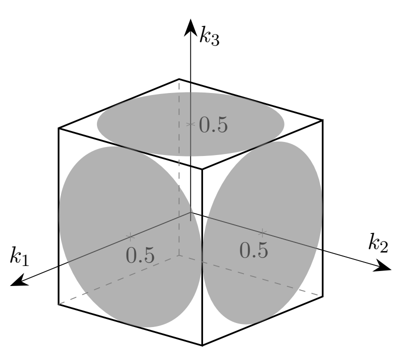
$$\Delta u + k_\pm^2 u = 0, \quad k_\pm = \omega/c_\pm, \quad \boldsymbol{x} \in \mathbb{R}^3,$$

where $c_\pm$ is the wave propagation speed in the host medium and inclusions, respectively. The solution u satisfies the transmission conditions (or Dirichlet or Neumann condition) on the boundary of inclusions and the Bloch condition at infinity: $\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}u(\boldsymbol{x})$ is periodic.

The local gap $\mathbb{G}(\boldsymbol{k}_0) = \mathbb{G}(\boldsymbol{k}_0, a)$ corresponding to $\boldsymbol{k} = \boldsymbol{k}_0$ consists of frequencies $\omega = \omega_a$ for which the following two conditions are satisfied:

(i) Bloch waves do not propagate when the wave vector $\boldsymbol{k}$ has the same direction as $\boldsymbol{k}_0$ and $|\boldsymbol{k} - \boldsymbol{k}_0|$ is sufficiently small. More precisely, Bloch waves with the frequency ω_a do not exist when $\boldsymbol{k} = (1+\delta)\boldsymbol{k}_0$ for sufficiently small *a-independent* $|\delta|$.

(ii) Bloch waves with $\omega = \omega_a$ propagate for some other $\boldsymbol{k}$ close to $\boldsymbol{k}_0$ (this condition will be specified).



As one of the consequences, it will be shown that for $\boldsymbol{k}$ on the boundary of the first Brillouin zone, the local gaps exist if and only if $|\boldsymbol{k}| < \sqrt{2}/2$ where the first Brillouin zone after an appropriate rescaling is the unit cube centered at the origin.

Shaded unit discs show the location of the Bloch vectors $\boldsymbol{k}_0$ on the boundary of the zone where local gaps $\mathbb{G}(\boldsymbol{k}_0, a)$ exist, i.e., Bloch waves do not propagate when $\boldsymbol{k} = (1+\delta)\boldsymbol{k}_0$, $\boldsymbol{k}_0$ is in the shaded region, $|\delta| < \delta_0$ with some $\delta_0 > 0$, and the time frequency ω is given by $\omega_0 + \mu_1 a^3 + \mathcal{O}(a^4) < \omega < \omega_0 + \mu_2 a^3 + \mathcal{O}(a^4)$, where $\omega_0 = c_+|\boldsymbol{k}_0|$ and there is an explicit formula for μ_i.

References

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